HYDRAULIC CONDUCTIVITY AND FLOW IN NON-UNIFORM SOIL

A.J. Peck and J.D. Watson

CSIRO Division of Land Resources Management
Private Bag, P.O.,
Wembley, W.A. 6014
Australia

Abstract

The paper examines the relationship between the frequency distribution \( F(K) \) of hydraulic conductivity measured from samples of volume \( V_1 \) and the bulk conductivity \( K_* \) of a volume \( V_2 \) \((V_2 >> V_1)\) of non-uniform homogeneous soil. A conclusive result is not reached, but it is argued that \( K_* \) probably lies between the geometric and arithmetic means of \( F(K) \). An expression is derived for estimating the effect of a given volume fraction of gravel stones on hydraulic conductivity.

Established heat flow theory is used to examine the contributions of root channels and lenses of permeable material to steady-state water flow in soils of relatively low hydraulic conductivity. These features are approximated by prolate and oblate spheroids oriented to maximize their contribution to flow. Although they provide no continuous path through the soil, the flow in these features can be very much greater than that in surrounding soil, and limited by the conductivity of material within the feature. In this sense they behave as continuous channels.
1. INTRODUCTION

Measurements of hydraulic conductivity in weathered rock aquifers in the Darling Range of Western Australia show essentially the same frequency distribution in each of four widely separated areas (Peck and Yendle, unpublished work). This should be no cause for surprise since the permeable strata are believed to be formed by the same pedogenetic processes operating on common parent material. But these results suggest that the sandy-clay material is non-uniform, homogenous in the sense discussed by Freeze (1975). That is, if measurements of hydraulic conductivity could be made on samples of increasing volume, a result similar to that sketched in Fig. 1 would be expected. Bear (1972) defines \( V_* \) as the representative elementary volume of the system. A set of measurements on samples with some volume \( V_1 < V_* \) will result in a range of values as indicated by the width of the band in this figure. Experience suggests that for any \( V_1 < V_* \) the frequency distribution of measured hydraulic conductivity can be reasonably well approximated by the log-normal function (Bouwer, 1969; Nielsen et al., 1973; Freeze, 1975).

Soil physicists normally measure hydraulic conductivity \( K \) by some form of in situ test, or by measurements on an "undisturbed" core sample. In each case the volume of the sample is usually much less than 1 m\(^3\). Yet the objective is often to characterize the soil on the scale of the spacing between tile drainage lines, or an elemental area of aquifer in a numerical study. The volume of soil involved in these applications may be several orders of magnitude larger than that sampled in the measurements.

In the first part of this discussion paper we examine the relationship between a frequency distribution of hydraulic conductivity \( F(K) \) measured on samples of volume \( V_1 < V_* \), and the bulk conductivity \( K_* \) which would be found on a sample of volume \( V_2 > V_* \). Childs (1969) and Freeze (1975) refer to this problem. Bouwer (1969) used a two-dimensional electric analog to determine \( K_* \) for two distribution functions of spatially random conductivity. He concluded that \( K_* \) lay between the harmonic and arithmetic means of \( F(K) \), and was reasonably well approximated by the geometric mean. Intuitively a somewhat different result is expected in three-dimensional flow because of the greater probability of bypass pathways around regions of relatively low conductivity.

The work referred to above considers a random spatial distribution of \( K \), but there is often a well-defined structure in soil resulting from pedogenesis, or activity of plants and animals. One approach to flow in such soils is to consider the structural elements alone. Bouma and Anderson (1973) have applied equations for flow in slits and tubes to estimate hydraulic conductivity from soil morphology. However, such channels and voids may be discontinuous and not inter-connected. In the second part of this paper we examine the contribution of what we call semi-continuous channels to flow through soil. The semi-continuous channel may be open at a boundary such as the soil surface, but it terminates within the soil mass without direct connection to any similar channel.
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Fig. 1. Schematic relationship between hydraulic conductivity $K$ and sample volume $V$ in a non-uniform, homogeneous soil. For any $V$ there is a frequency distribution $F(K)$ of values of $K$. $V_*$ is the presentative elementary volume.

2. INCLUSION MODELS

Analysis of the effect of a body of one conductivity embedded in a more or less conductive medium has been valuable in relating the thermal conductivity of a granular material such as soil to the properties and volume fractions of its components (deVries, 1963). In this section we apply similar methods to examine the analogous hydraulic problem.

2.1 Single Spherical Inclusion

Consider an infinite medium of conductivity $K_0$ in which there is a uniform potential gradient $G$. Carslaw and Jaeger (1959) discuss the distortion of the potential field which results when a body of conductivity $K_1$ is embedded in the medium. In the case of a spherical body of radius centered on the origin of the coordinate system, it is shown that the steady-state potential gradient $G_s$ at a point $r$ is

$$ G_s = \begin{cases} G(1 + a^3 K_o - K_1)/r^3(2K_o + K_1) & \text{if} \ r > a \\ 3G K_o/(2K_o + K_1) & \text{if} \ r < a \end{cases} \quad (1) $$

Equation (1) can be used to calculate the uniform conductivity $K_*$ of a sphere of radius $R$ which would provide the same total flux through the surface $r=R$ as would the sphere of radius $a$ ($a<R$) and conductivity $K_1$ embedded in the medium of conductivity $K_o$:

$$ \frac{K_*}{K_o} = \frac{(1+2a) K_o}{(1-a) K_o + 2 + a} \cdot (3) $$

In this equation $\alpha = a^3/R^3$ represents the volume fraction of the included material. By superposition, equation (3) also applies when many, non-interacting spheres of the same conductivity make up the total volume faction of included material (Carslaw and Jaeger, 1959).

Figure 2 shows equation (3) graphed in the form $K_*/K_o$ as a function of $K_1/K_o$ for several values of the parameter $\alpha$. Asymptotic values of $K_*/K_o$ follow from equation (3): $K_*/K_o \to 2(1-\alpha)/(2+\alpha)$ as $K_1/K_o \to 0$, and $K_*/K_o \to (1+2\alpha)/(1-\alpha)$ as $K_1/K_o \to \infty$. 

Figure 2 shows that if spherical stones with negligible conductivity ($K_1=0$) are included in a permeable matrix, the conductivity would be reduced by 33% for $\alpha = 0.25$, or by 7% for $\alpha = 0.05$. In each case it is assumed that the stones are so far apart that they behave independently. Interaction between intruded bodies could be expected to further reduce the conductivity of the heterogeneous medium.

![Figure 2. Bulk conductivity $K_*$ of a medium with conductivity $K_0$ which there is a volume fraction $\alpha$ of non-interacting spherical inclusions with conductivity $K_1$.](image)

The case of the included sphere with conductivity much greater than the background does not appear to have application in soil physics. Even within a spherical cavity in a saturated soil the flux density is only increased by a factor 3 from its value in the background medium. This result follows from equation (2).

### 2.2 Effect of Many Inclusions

deVries (1963) has considered the thermal conductivity of a medium containing a great number of randomly oriented ellipsoidal inclusions. Many measurements of the thermal conductivity of granular media such as soils are available for comparison with estimates from theory.

If $K_0$ is the conductivity of the background medium, volume fraction $\alpha_0$, and $K_1, K_2, \ldots K_N$ the conductivities of spherical inclusions with volume fractions $\alpha_1, \alpha_2, \ldots \alpha_N$, then deVries' expression reduces to

$$K_* = \frac{\sum_{i=0}^{N} k_i \alpha_i K_i}{\sum_{i=0}^{N} k_i \alpha_i}$$  \hspace{1cm} (4)

where

$$k_i = \frac{3K_0}{(2K_0 + K_i)}.$$
In the case of a two-component system, equation (4) reduces to equation 3.

Although equation (4) is developed for situations where there is no interaction between the included spheres, deVries (1963) reports good agreement between computed and measured thermal conductivities for soils of varying mineral composition and moisture content. He claims that the difference between a value predicted by equation (4) and the true thermal conductivity of a two-component system is less than 10% for $0 < K_1/K < 10$. When $K_1/K_0$ is of order 100, computed values are about 25% too low through the range $0.4 < \alpha < 0.7$.

It appears that equation (3) and Fig. 2 should be applicable with reasonable confidence to estimate the effective hydraulic conductivity of media such as gravelly or stony soils when the volume fraction of the stones and the hydraulic conductivity of the matrix material are known. Moreover, equation (4) appears to be applicable to systems which can be regarded as an assembledge of components of different hydraulic conductivity.

### 2.3 Continuous Distribution Theory

Equation (4) can be generalised to a continuous distribution of spherical inclusions with volume fraction $F(K) dK$ and conductivity $K$:

$$K_* = \frac{\int [KF(K)/(2K+K)] dK}{\int [F(K)/(2K+K)] dK}.$$  \hspace{1cm} (5)

Empirical evidence (Bouwer, 1969; Nielsen et al., 1973; Freeze, 1975) suggests that the frequency distribution of hydraulic conductivity in a region is closely approximated by the log-normal function

$$F(K) = \frac{1}{a\sqrt{2\pi}} \exp\left[-\frac{(\ln K - \mu)^2}{2\sigma^2}\right].$$  \hspace{1cm} (6)

where $\mu$ and $\sigma^2$ are the true mean and variance which may be estimated by

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \ln K_i,$$

$$\sigma^2 = \frac{1}{(N-1)} \sum_{i=1}^{N} (\ln K_i - \mu)^2.$$

The log-normal frequency distribution of equation (6) may be substituted in equation (5) to determine $K_*$ given some $K_0$.

When $K_0=0$ it is readily shown that

$$K_* = \exp(\mu-\sigma^2/2)$$  \hspace{1cm} (7)

which is the harmonic mean $K_h$ approximated by

$$K_h = \frac{N}{\sum_{i=1}^{N} (1/K_i)}.$$
The harmonic mean is also found to be the bulk conductivity when blocks of material with conductivity $K$ and frequency distribution $F(K)$ are assembled purely in series. In terms of the granular medium, it appears that the assumption $K_0 = 0$ is equivalent to assuming that all flow takes place through the included material and in the direction of the global potential gradient.

If $K_0 = \infty$ it is found by substitution of $F(K)$ from equation (6) into equation (5) that

$$K_* = \exp (\mu + \sigma^2/2).$$

This is the arithmetic mean $K_a$ approximated by

$$K_a = \frac{1}{N} \sum_{i=1}^{N} K_i.$$

The arithmetic mean represents bulk conductivity when the soil behaves as a system with flow through parallel resistances. In terms of the granular medium, the assumption $K_0 = \infty$ implies that the fluid itself is the background medium.

Thus we are able to conclude that the bulk conductivity $K_*$ will lie somewhere between the harmonic mean $K_h$ and the arithmetic mean $K_a$ when $F(K)$ is log-normal. Unfortunately this is no advance on the conclusion reached by Bouwer (1969) from his two-dimensional analog study. Since his investigation showed that the geometric mean $K_g$ was a reasonable estimate of $K_*$, we suspect that in systems allowing three-dimensional internal flow, $K_*$ will lie somewhere between $K_g$ and $K_a$.

3. FLOW IN RELATIVELY PERMEABLE CHANNELS AND LENSES

Semi-continuous features such as old root channels and lenses of relatively coarse material are often observed in soil profiles. The contribution of these components to flow of water through soil has long been discussed in qualitative terms. In this section we examine the effects of semi-continuous tubular channels and planar voids by approximating their geometries with ellipsoids with semi-axes $a$, $b$ and $c$. Quite clearly the effect of these features will depend on their orientation relative to that of the global potential gradient. We have considered only two cases.

3.1 Flow in a Root Channel

The single root channel is approximated by a prolate spheroid $a>b=c$ of material with conductivity $K_1$ embedded in a medium with conductivity $K_0$. We consider the case where the undistorted potential gradient is in the $x$-direction. This is the direction of $a$, which represents the length of the root. In this orientation the root channel will make its greatest contribution to flow through the soil as a whole.

Carslaw and Jaeger (1959) present an expression for the potential gradient $G_p$ within the prolate spheroid in the steady-state:

$$G_p = G/\left[1 + A_0 (K_1/K_0 -1)\right]$$

(9)
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where $G$ is the global potential gradient, and for $a >> b = c$

$$A_a = (c^2/a^2) (\ln 2a/c - 1).$$

We define $q$ as the ratio of the flux density within the spheroid to that through the undisturbed background medium. Then it follows from equation (9) that

$$q = \frac{(K_1/K_0)}{[1 + A_a (K_1/K_0 - 1)]},$$

(10)

and $q \rightarrow A_a^{-1}$ as $K_1/K_0 \rightarrow \infty$.

The broken lines in Fig. 3 show equation (10) in the form $q$ as a function of $K_1/K_0$ for various values of the geometric parameter $a/c$. The solid curve for $a/c = 1$ presents results for the sphere which can be derived from equation (2).

The ratio $q$ of fluxes within and distant from the prolate spheroid which represents a root channel, approaches, but is always less than the ratio of the conductivities of the media $K_1/K_0$. This difference can be substantial in channels which are short relative to their radius ($a/c$ small), reflecting a limitation on flow through the root channel by its surface area and the conductivity of the surrounding medium.

When root channels penetrate clay soils, lateritic duricrusts, silicified strata etc. it is conceivable that $K_1/K_0$ may take values up to $10^4$ or more, and $a/c$ may approach $10^3$. Then it will be seen from Fig. 3 that the flux within the channel is primarily controlled by the conductivity of the channel material ($q = K_1/K_0$). It follows that a relatively small spatial density of channels containing high-conductivity material can make a substantial contribution to the total flow of water through soils of very low conductivity. It may be advisable in field studies to determine the spatial density, dimensions and conductivity of channels such as old root paths as well as the conductivity of the background material. Bouma and Anderson (1973) emphasize the importance of channels, which reflect soil morphology, in water transport on a scale of order 0.1 m.

![Fig. 3. Ratio of flux densities within and distant from ellipsoidal inclusions. Broken lines represent prolate ($a>b=c$) and solid lines oblate ($a=b>c$) forms. The parameter is $a/c$. See text for other details.](image-url)
3.2 Flow in a Lens of Permeable Material

A lens or planar distribution of material of conductivity $K_1$ in soil of conductivity $K_0$ can be approximated by the oblate spheroid $a=b>>c$. We consider the case where the undistorted potential gradient is in the $x$-direction so that the ellipsoidal inclusion makes the greatest contribution to flow. The distortion of the potential field is reported by Carslaw and Jaeger (1959). Following the terminology of the previous section, we find in this case

$$q = 4K_1/K_0 \left[ 4 + \pi (K_1/K_0 - 1) c/a \right]$$

(11)

and $q\rightarrow4a/\pi c$ as $K_1/K_0 \rightarrow \infty$.

The solid lines in Fig. 3 show $q$ as a function of $K_1/K_0$ with $a/c$ as the parameter. Comparison with the results for the prolate spheroid (the broken lines) show that for the same ratio of conductivities $K_1/K_0$ and geometric factor $a/c$ the flux ratio is less for the planar channel (oblate spheroid) than for the root channel. This reflects the difference between the basically cylindrical flow from the root channel, and the more nearly one-dimensional flow away from the planar channel in the orientations considered. A semi-continuous planar channel will be less effective in conducting water through a soil of relatively low conductivity than a root channel with the same cross-sectional area.

4. GENERAL DISCUSSION

The estimation of $K_*$ given some frequency distribution $F(K)$ of conductivity which is randomly distributed in space remains a significant problem. When $K$ is measured on a scale of $d^3$ m$^3$, and flow is normal to a plane of the material with area $A$ ($>d^2$) and thickness $d$, then clearly the arithmetic mean $K_*$ will be the best estimate of $K_*$. But when flow is in the plane, we have Bouwer's conclusion that the geometric mean $K_g$ is a good estimate of $K_*$. The uncertainty arises when each of the aquifer dimensions is much greater than $d$. In view of the labour involved in the construction of a three-dimensional electric analog of this problem, it may be preferable to solve the appropriate network equations using a digital computer. Our preliminary investigation suggests that this should be a relatively straightforward task.

Our results for prolate and oblate spheroids depend on the geometric ratio $a/c$, but not on the absolute size of the channel or void. Therefore they will apply equally well to channels left by very small roots and voids between peds to those left by very large tree roots, lenses of sand in sedimentary material etc. In this respect we note that when $a/c$ is relatively large, flow into the channel or void may be essentially limited by $K_1$. That is, the rate of flow into the channel is not significantly reduced by the fact that it terminates within soil of relatively low conductivity.

Finally we emphasize that the effects of channels and voids computed in this paper apply to steady state flow whether it be saturated or not. Undoubtedly these structures are also important in transient soil-water phenomena such as infiltration, and this would appear to be a valuable area for further study. The present approach could be developed to estimate the lower bound on infiltration into soils containing channels or voids.
5. REFERENCES


