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CLASSICAL THEORY OF DECOMPRESSION AND THE DESIGN OF SCUBA DIVING TABLES

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Abstract

We summarise the classical (gas diffusion) theory of decompression, which is an interesting application of elementary differential equations. We show that the derivation of recreational scuba diving tables from this theory is an ill-defined problem in optimisation.

Keywords: compartment models; Dalton's Law; diffusion; Fick's Law; Lagrange multipliers; optimisation; ordinary differential equations.

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1. Introduction

All scuba divers are trained to follow procedures that limit the time spent at depth, in order to minimise the risk of decompression sickness. These procedures, encoded either on waterproof tables or in the algorithm of a diving computer, are based on a mathematical theory that is relatively simple in structure. Classical decompression theory, first developed by J.S. Haldane [4, 6], describes the diffusion of nitrogen in the diver's body by a simple independent-compartment model, consisting of a small system of ordinary differential equations governed by some simple constraints.

As millions of young people have now been certified as scuba divers [9, p. 32],

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educators may find it useful to give decompression theory as a motivating example in calculus classes. We shall see that much of the analysis requires little more than careful consideration of the solutions of relatively straightforward differential systems.

In addition to explaining classical decompression theory, we aim to show how the design of optimal diving procedures is a rather ill-defined process which is worthy of further study. A day of recreational diving often involves successive dives, to different depths, separated by a rest period at the surface (known as the surface interval). The design of recreational diving tables can be formulated as an optimisation problem. Existing tables are often protected by commercial secrecy, and we are not aware of any detailed discussion of table design in the open literature.

The remainder of the article is set out as follows. In §2 we consider some of the basic theory behind the constraints that limit the duration of a dive. This sets the parameters of the problem and enables the subsequent calculations to be considered in the following sections. We investigate two relatively simple cases; in §3 we examine the limits to the duration of a single no-decompression dive. This analysis is extended in §4 to considering the optimal plan for two dives separated by a prescribed surface interval. We close with a short discussion.

2. The basic theory

The definitive modern text on diving physiology and medicine is that by Bennett & Elliott [5]; decompression theory is covered in a chapter by Tikvisis & Gerth [10] which should be consulted for technical references. Accessible, popular introductions to diving physiology are given by Bookspan [3], Martin [8] and Lippmann [7]. What follows is a simplified summary.

Scuba equipment is designed to deliver the breathing air to the diver at a pressure equal to that of the ambient water. The classical explanation for decompression sickness [2] is that, at this increased pressure, nitrogen in the breathing air diffuses into the diver's body. When the diver returns to the surface, the ambient pressure is reduced and the body now contains an excess of nitrogen. Nitrogen can diffuse from the body tissues back into the lungs where it is exhaled. However if the quantity of dissolved nitrogen in a tissue exceeds a certain critical value, nitrogen will come out of solution and form bubbles of nitrogen gas, either in the blood or in other tissues. The presence of these bubbles leads to decompression sickness and other ills. For a popular demonstration, take two bottles of carbonated soft drink: opening the first bottle rapidly leads to the formation of large bubbles, while opening the second bottle slowly does not.

2.1. The diffusion of nitrogen

The ambient water pressure p increases linearly with depth so at a distance d beneath the surface it is $p = p_0 + \lambda d$ atmospheres, where p_0 is the pressure at the water surface. Typically $p_0 = 1$ atmosphere at sea level and λ is approximately 0.1 atmospheres per metre in sea water; in other words at a depth of 10 metres the water pressure is roughly twice the sea-level surface pressure p_0 and increases by one atmosphere for every further 10 metres descended. Since air is composed of approximately 21% oxygen and 79% nitrogen, the nitrogen component of the diver's breathing air exerts a partial pressure of $0.79p$ atmospheres according to Dalton's Law. More generally we may consider dives in lakes at higher altitudes (that is a reduced surface pressure p_0) or using a breathing gas which has a different fraction μ of nitrogen. Naturally the partial pressure of nitrogen in the breathing gas is then simply μp.

Nitrogen diffuses from the breathing gas in the lung air space into the diver's blood, and then in turn, into other bodily tissues. The diffusion from air into blood occurs so quickly that it can be treated as a virtually instantaneous process and the blood then effectively also carries a nitrogen partial pressure of μp atmospheres. This diffuses slowly into other bodily tissues and in the classical theory it is assumed there are a finite number of these, say m, each connected independently to the blood. If we label these tissues i with $i = 1, \ldots, m$, and suppose that the nitrogen saturation (or tension) in tissue i is x_i atmospheres, then diffusion is assumed to satisfy Fick's law. This asserts that the rate of diffusion of nitrogen from the blood into the tissue is proportional to the difference in their respective nitrogen concentrations. Mathematically, this means that the time derivative of nitrogen tension is simply

$$\dot{x}_i = -k_i(x_i - \mu p) \tag{1}$$

where the value of the diffusion constant k_i depends on the characteristics of tissue i. Thus the complete state of the body can be summarised in the form of a vector $\mathbf{x} = (x_1, \ldots, x_m)$ of tissue nitrogen tensions governed by independent diffusion equations.

The constants k_i in equation (1) are usually quoted in the literature in terms of 'half-times' τ_i defined to be the time elapsed for an initial saturation $x_i = x_i(0)$ to be reduced by a factor two when the surrounding ambient pressure $p = 0$. It is easy to show that $\tau_i = \ln(2)/k_i$; for human tissues these half-times typically lie in the range from about 10 minutes to as long as 6 hours. The original model formulated by Haldane [4, 6] consisted of five tissues (or compartments) with halftimes between 5 and 75 minutes; a more up-to-date example with eight compartments is the DSAT model with parameters as listed in Table 1.

compartment i	1	2	3	4	5	6	7	8
halftime τ_i (minutes)	5	10	20	30	40	60	80	120
saturation $M_{0,i}$ (atm)	3.035	2.533	2.049	1.830	1.707	1.576	1.507	1.438

TABLE 1: Typical halftime and surfacing M-values for the eight compartments used in the DSAT model.

Classical decompression theory is founded on the assumption that each tissue i can tolerate a maximum nitrogen tension of $M_i(p)$ before nitrogen bubbles begin to form. This critical tension, commonly referred to as the M-value, is a function of the ambient pressure p and increases with p. In the original Haldane model [4] it was assumed that, for a diver breathing compressed air, the critical nitrogen tension is simply twice the partial pressure of nitrogen in the breathing air at ambient pressure; this then implies that $M_i(p) = 1.58p$ for all compartments i. In more sophisticated models, the critical nitrogen tension is taken to be a linear function of p with coefficients that depend on the properties of the particular tissue under investigation. In this case

$$M_i(p) = M_{0,i} + (p-1)M_i' \tag{2}$$

where $M_{0,i}$ is the maximum nitrogen saturation that can be tolerated at a pressure of one atmosphere; this is often known as the “surfacing M-value” when the water surface is at sea level. Table 1 includes representative surfacing M-values extracted from Bookspan [3, pp. 16, 23].

2.2. Dive planning constraints

Before descending it is normal practice for the diving party to agree on the duration and purpose of the impending dive. This ‘dive plan’ specifies the intended depth $d(t)$ as a function of time t during a dive and generally is one of two basic kinds. The first, a *no-decompression* dive, is one which, in theory, can be aborted at any time without requiring special decompression procedures. For this to be possible, it has to be ensured that the nitrogen saturation in each of the diver’s tissue compartments never exceeds the maximum nitrogen saturation that can be tolerated at sea level: this demands that

$$x_i(t) \leq M_i(p_0) \quad \text{for all } t. \tag{3}$$

In contrast, on a *decompression* dive, the diver may not be able to ascend immediately to the surface at any time: obligatory decompression stops are required during the ascent. This more complicated type of dive plan satisfies only the minimal requirement for avoiding decompression sickness that the nitrogen saturation in each tissue is less than the maximum saturation appropriate to the present depth. This then imposes the weaker requirement that

$$x_i(t) \leq M_i(p(t)) = M_i(p_0 + \lambda\mu d(t)) \quad \text{for all } t \tag{4}$$

and, as we have already remarked that $M_i(p)$ is frequently taken to be a linear function of p, the constraint (4) becomes linear in depth so

$$x_i(t) \leq (M_{0,i} + (p_0 - 1)M'_i) + \lambda\mu M'_i d(t) \quad \text{for all } t. \tag{5}$$

It is worth noting that the Haldane model was developed and used for planning staged decompression dives in military and commercial circumstances, rather than no-decompression dives which are more common for recreational purposes.

3. No-decompression limits

3.1. Theory

A simple optimization problem is to determine the “no-decompression limit” for a given depth D. Put simply, this is the maximum permissible duration T of a no-decompression dive to a constant depth D, starting and ending at the surface. To a first approximation it may be assumed that ascent and descent occurs quickly, so that the dive profile can be idealised as a simple square form, $d(t) = D$ if $0 < t < T$ and $d(t) = 0$ otherwise. For the purposes of the analysis it is assumed that the diver is fresh, that is, they have not had recent exposure to changes in pressure; their initial state $\mathbf{x}(0)$ is in equilibrium with air at sea level. Then $x_i(0) = 0.79$ atmospheres for all compartments i although for planning multiple dives during a day the no-decompression limit for a diver is critically dependent on their particular recent diving history.

Thus we seek to maximise T subject to the constraints

$$x_i(t) \leq M_i(p_0) \quad \text{for all } t$$

where

$$\dot{x}_i = -k_i(x_i - \mu p(t))$$

with initial condition $x_i(0) = 0.79$ unless otherwise stated. Writing $P = p_0 + \lambda D$ for the ambient pressure at depth D, and $Q = \mu P = \mu p_0 + \mu\lambda D$ for the nitrogen partial pressure at this depth, we have

$$\dot{x}_i = -k_i x_i + k_i Q.$$

This first order DE has solution

$$x_i = Q + A_i e^{-k_i t}$$

where the constant A_i is determined by the initial value $x_i(0)$ yielding

$$x_i(t) = Q + (x_i(0) - Q)e^{-k_i t}. \tag{6}$$

Since $x_i(t)$ is monotonically increasing, the maximum permitted nitrogen tension $M_i(p_0)$ is reached at time

$$T_i = -\frac{1}{k_i} \ln \left[\frac{Q - M_i(p_0)}{Q - x_i(0)} \right] \tag{7}$$

provided $Q \geq M_i(p_0)$; otherwise this limit is never reached.

Thus, let $\textsc{ndl}(D, \mathbf{x})$ be the no-decompression time limit for a square dive to depth D for a diver with initial state vector $\mathbf{x}$. Then we have shown that

$$\textsc{ndl}(D, \mathbf{x}) = -\min_i \frac{1}{k_i} \ln \left[\frac{Q - M_i(p_0)}{Q - x_i(0)} \right] \tag{8}$$

where the minimum is taken over those i such that $M_i(p_0) \leq Q$, where $Q = \mu p_0 + \mu\lambda D$.

In particular, for a dive at sea level ($p_0 = 1$) on air ($\mu = 0.79$) by a fresh diver ($x_i(0) = 0.79$) we have $M_i(p_0) = M_{0,i}$, $Q = \mu(1 + \lambda D)$ and $Q - x_i(0) = \mu\lambda D$ in (8).

We define the *'controlling tissue'* for a no-decompression dive to be the tissue index i which achieves the minimum in (8).

3.2. Calculations

The no-decompression limits depend crucially on the behaviour of the surfacing M-values. In the Haldane model, $M_{0,i} = 1.58$ for all i, so that for a fresh diver, the expression

$$\ln \left[\frac{Q - M_i(p_0)}{Q - x_i(0)} \right]$$

is equal for all i. Hence the minimum in (8) is achieved by taking k_i as large as possible, and the no-decompression limit for a fresh diver (at sea level using compressed air) is always dictated by the compartment with the shortest halftime. The resulting no-decompression limits are shown in Table 2.

Depth (m)	NDL (minutes)		Controlling tissue i
	Haldane	DSAT	
10	∞	275.5	7
12	12.9	152.8	6
14	9.0	102.0	5
16	7.1	74.6	5
18	5.8	56.9	4
20	5.0	46.0	3
22	4.4	37.2	3
24	3.9	31.5	3
26	3.5	27.2	2
28	3.2	22.4	2
30	2.9	19.2	2

TABLE 2: Comparison of theoretical no-decompression limits (NDL) under the Haldane and DSAT models and calculated using (8). The rightmost column identifies the tissue compartment i in the DSAT model which fixes the NDL. The calculations assume that the diver is fresh, breathes compressed air, and the surface is at sea-level.

For the DSAT model (given in Table 1), the surfacing M-values decrease with i, and range from $M_{0,1} = 3.03 > 1.58$ for the fastest compartment to $M_{0,8} = 1.43 < 1.58$ for the slowest. These lead to no-decompression limits (NDLs) which are much longer than the Haldane predictions and, perhaps surprisingly, it is not always the fastest compartment which fixes the upper limit on the duration of the dive. The sample results in Table 2 suggest that at relatively shallow depths the slow compartments determine the length of the dive and the expected importance of the faster compartments only becomes apparent for deeper excursions.

It is easy to show that the NDL is a decreasing function of depth and much of its attraction lies in the conservative assumptions applied for its calculation. We have mentioned already that when finding the NDL the diver is supposed to reach his maximum depth D immediately, and stay at that depth for the entire duration of the dive. In practice of course the diver cannot descend and ascend instantaneously but it can be shown that the set of all no-decompression dives is 'monotone' in the sense that if $d_1(t)$ is a no-decompression dive and $d_2(t) \leq d_1(t)$ for all t, then $d_2(t)$ is also a no-decompression dive. Hence any dive with a maximum depth of D and maximum duration $t \leq \text{NDL}(D)$ is guaranteed to be a no-decompression one and the diver can proceed safe in the knowledge that decompression will not be required.

It may be of interest to determine how the predicted NDL is affected should the descent and/or ascent be specified. Divers frequently change from one depth to another at a steady speed and it turns out that the governing equation for $x_i(t)$ can still be solved explicitly if the depth is taken to be a linear function of time. Further elaboration can be introduced by assuming that the diver is not fresh and has dived previously on the same day and we consider this now.

4. Optimal planning for two no-decompression dives

4.1. Double dives

Recreational divers normally plan two dives for the day separated by a period at the surface. Suppose the first dive has a square profile to a depth d_1 metres for time t_1 minutes, followed by a surface interval of s minutes, and then a second dive which is also a square profile to depth d_2 for time t_2 minutes. This simple profile is illustrated in Figure 1.

In all that follows we shall assume that the depths d_1, d_2 and the surface interval s are all fixed, and the task is to optimise (some function of) the dive durations t_1, t_2 subject to the no-decompression constraint.

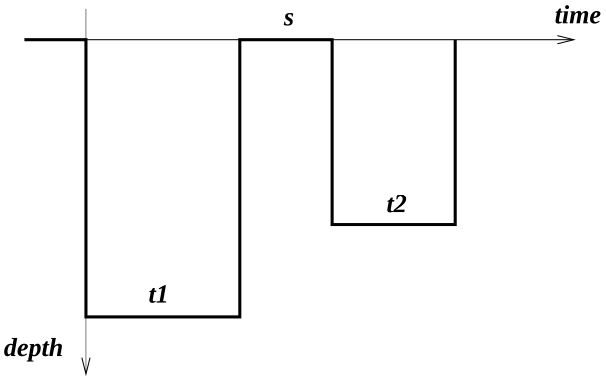

FIGURE 1: A typical double dive plan profile

Let us assume that the diver begins fresh; then by equation (6), on surfacing from the first dive, the diver has tissue states

$$x_i(t_1) = \mu p_0 + \mu\lambda d_1(1 - e^{-k_i t_1})\,.$$

While back on the surface, tissues desaturate according to

$$\dot{x}_i = -k_i x_i + k_i \mu p_0$$

so that at the end of the surface interval of s minutes and immediately before the second dive,

$$\begin{aligned} x_i(t_1 + s) &= x_i(t_1)e^{-k_i s} + \mu p_0(1 - e^{-k_i s}) \\ &= [\mu p_0 + \mu\lambda d_1(1 - e^{-k_i t_1})]e^{-k_i s} + \mu p_0(1 - e^{-k_i s}) \\ &= \mu p_0 + \mu\lambda d_1(1 - e^{-k_i t_1})e^{-k_i s}. \end{aligned} \tag{9}$$

During the second dive, tissues again take in nitrogen, according to

$$\dot{x}_i + k_i x_i = k_i \mu(p_0 + \lambda d_2)$$

with general solution

$$x_i = C_i e^{-k_i t} + \mu(p_0 + \lambda d_2)$$

so that on surfacing again

$$\begin{aligned} x_i(t_1+s+t_2) &= x_i(t_1+s)e^{-k_i t_2} + \mu(p_0+\lambda d_2)(1-e^{-k_i t_2}) \\ &= [\mu p_0 + \mu\lambda d_1(1-e^{-k_i t_1})e^{-k_i s}]e^{-k_i t_2} + \mu(p_0+\lambda d_2)(1-e^{-k_i t_2}) \\ &= \mu p_0 + \mu\lambda d_1(1-e^{-k_i t_1})e^{-k_i(s+t_2)} + \mu\lambda d_2(1-e^{-k_i t_2})\,. \end{aligned}$$

Thus the final tissue saturation is

$$x_i(t_1+t_2+s) = \mu p_0 + \mu\lambda d_1(1-e^{-k_i t_1})e^{-k_i(s+t_2)} + \mu\lambda d_2(1-e^{-k_i t_2}). \qquad (10)$$

If we define

$$Q_0 = \mu p_0, \qquad Q_1 = \mu\lambda d_1 \quad \text{and} \quad Q_2 = \mu\lambda d_2$$

then these results can be rewritten as

$$\begin{aligned} x_i(t_1) &= Q_0 + Q_1 - Q_1 e^{-k_i t_1} \\ x_i(t_1+t_2+s) &= Q_0 + Q_2 + (Q_1 e^{-k_i s} - Q_2)e^{-k_i t_2} - Q_1 e^{-k_i(t_1+t_2+s)}. \end{aligned}$$

To satisfy the no-decompression requirement (3) for all $0 \le t \le t_1+t_2+s$, it suffices (by monotonicity) to simply apply this constraint at the ends of each of the two dives, i.e. to demand that

$$x_i(t_1) \le M_{0,i} \qquad \text{and} \quad x_i(t_1+t_2+s) \le M_{0,i}.$$

The first inequality just forces $t_1 \le \textsc{ndl}(d_1)$, the no-decompression limit for the depth d_1. For a fixed $t_1 \le \textsc{ndl}(d_1)$, the second constraint is then satisfied in the interval $t_2 \in [0, T_2]$ where $T_2 = g(t_1) = \textsc{ndl}(d_2, \mathbf{x}(t_1+s))$ is the no-decompression limit for a diver with initial state (9). Hence the set of solutions (t_1, t_2) that together satisfy our two constraints is the subgraph

$$S = \{(t_1, t_2) : \; 0 \leq t_2 \leq g(t_1), \; 0 \leq t_1 \leq T_1\}$$

where $T_1 = \textsc{ndl}(d_1)$. The boundary is piecewise differentiable, with knots occurring where the controlling tissue changes (i.e. where the minimum in (8) with $\mathbf{x} = \mathbf{x}(t_1 + s)$ is achieved by two different tissues i).

Figure 2 shows an example solution set for the choices of $d_1 = 40$ metres, $d_2 = 12$ metres and $s = 15$ minutes. As might have been expected, the boundary is approximately linear with a negative slope steeper than unity (left panel). Closer inspection (right panel) reveals that the boundary is convex and slightly nonlinear.

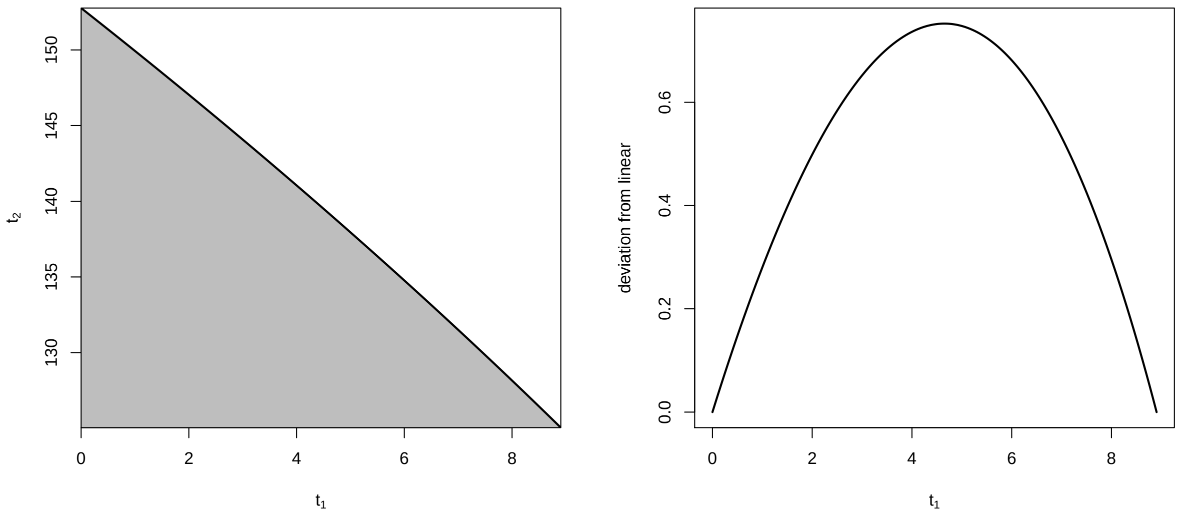

FIGURE 2: Example of solution set for double dive. *Left:* Maximum permissible duration of the second dive, t_2, as a function of the duration of the first dive t_1. *Right:* Deviation of the boundary from a straight line. Here the parameters are $d_1 = 40$ metres, $d_2 = 12$ metres, $s = 15$ minutes and the times are computed using the DSAT model.

4.2. The optimal double dive

We now consider how one might attempt to design the most desirable double dive. Of course before much progress can be made some appropriate objective function must be defined that measures the attractiveness of any particular dive plan. In the absence of any other function, an obvious start point would be to consider the total time $t_1 + t_2$ spent underwater. In that case the problem reduces to maximising $T = t_1 + t_2$ subject to

$$\begin{aligned} Q_0 + Q_1 - Q_1 e^{-k_i t_1} &\leq M_{0,i} \qquad (11) \\ Q_0 + Q_2 + (Q_1 e^{-k_i s} - Q_2) e^{-k_i t_2} - Q_1 e^{-k_i (t_1 + t_2 + s)} &\leq M_{0,i} \qquad (12) \end{aligned}$$

for all i. However it can easily be shown that the solution is trivial: if $d_1 > d_2$ then the solution is just $t_1 = 0$ and $t_2 = \text{NDL}(d_2)$. This result is intuitively obvious when it is realised that the deeper a diver goes so the build-up in nitrogen concentration progressively increases. Thus to spend the maximum total time the diver simply elects to go to the shallower depth for as long as the no-decompression limits allow, and does not dive the deeper depth at all.

This result implies that maximising the total dive time is not a sensible measure of a good dive plan. Most divers will attest that deep dives are in some sense more exhilarating and fulfilling than shallow ones; so as a refined objective function, let us look to the integral of depth over time

$$\Phi = t_1 d_1 + t_2 d_2$$

subject to the constraints (11) and (12).

To calculate the maximum value of the objective function Φ we use the technique of Lagrange multipliers. We therefore introduce constants $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$

and define

$$Y = d_1 t_1 + d_2 t_2 + \sum_{i=1}^{m} A_i a_i(t_1) + \sum_{i=1}^{m} B_i b_i(t_1, t_2) \tag{13}$$

where

$$\begin{aligned} a_i(t_1) &= M_{0,i} - Q_0 - Q_1 + Q_1 e^{-k_i t_1} \\ b_i(t_1, t_2) &= M_{0,i} - Q_0 - Q_1 e^{-k_i s}(1 - e^{-k_i t_1}) e^{-k_i t_2} - Q_2(1 - e^{-k_i t_2}). \end{aligned}$$

are the constraint functions corresponding to (11) and (12) respectively.

The optimal solution may occur either at

(a) a boundary point of the time domain, or at

(b) a generic stationary point of Y, or at

(c) a point where two of the constraint functions are equal to zero.

Case (a) We must consider the boundary solutions occurring when either $t_1 = 0$ or $t_2 = 0$ corresponding to only a single dive. The possible boundary points are then

(a1) $(t_1, t_2) = (0, \text{NDL}(d_2))$ and

(a2) $(t_1, t_2) = (\text{NDL}(d_1), 0)$

but case (a2) can be shown to be suboptimal as follows. For a fixed value of t_1, the value of $d_1 t_1 + d_2 t_2$ is clearly greatest when t_2 is maximised subject to the constraints. This is to say that t_2 should equal $\text{NDL}(d_2, \mathbf{x}(t_1 + s))$ with $\mathbf{x}(t_1 + s)$ given by (9). If $t_1 = \text{NDL}(d_1)$ then, for any nonzero surface interval s, the tissue saturations $\mathbf{x}(t_1 + s)$ at the start of the second dive are clearly sub-critical, so that $\text{NDL}(d_2, \mathbf{x}(t_1 + s)) > 0$, and sub-case (a2) is not optimal, and can be safely excluded from further consideration.

Case (b) Stationary points of Y are found by considering the partial derivatives

$$\begin{aligned}\frac{\partial Y}{\partial t_1} &= d_1 - Q_1 \sum_i A_i k_i e^{-k_i t_1} - Q_1 \sum_i B_i k_i e^{-k_i(s+t_1+t_2)} \\ &= d_1 - Q_1 \sum_i A_i k_i e^{-k_i t_1} - \sum_i B_i k_i Q_{1,i} e^{-k_i(t_1+t_2)} \qquad (14) \\ \frac{\partial Y}{\partial t_2} &= d_2 + \sum_i B_i k_i \left[Q_1 e^{-k_i(s+t_2)}(1-e^{-k_i t_1}) - Q_2 e^{-k_i t_2} \right] \\ &= d_2 + \sum_i B_i k_i e^{-k_i t_2} \left[Q_{1i}(1-e^{-k_i t_1}) - Q_2 \right] \qquad (15)\end{aligned}$$

where we have written $Q_{1,i} = Q_1 e^{-k_i s}$ since the surface interval duration s is taken to be fixed.

Generic stationary points of (13) are determined by solving

$$\frac{\partial Y}{\partial t_1} = \frac{\partial Y}{\partial t_2} = 0$$

subject to the additional requirement that one of the constraint functions has value zero (while the other constraint functions are all positive). There are essentially two sub-cases:

(b1) $a_i(t_1) = 0$ for some i whereupon the corresponding Lagrange multiplier A_i is nonzero while all other constraint functions are positive and their Lagrange multipliers zero.

(b2) $b_i(t_1, t_2) = 0$ for some i. Now B_i is nonzero with all the other constraint functions positive with corresponding Lagrange multipliers zero.

Sub-case (b1) implies that t_1 is equal to the no-decompression limit $\textsc{ndl}(d_1)$ and that t_2 is strictly less than the resulting no-decompression limit $\textsc{ndl}(d_2, \mathbf{x}(t_1+s))$ for the second dive. However, this combination is suboptimal and can be excluded by the same argument used to dismiss (a2).

Thus we need only consider sub-case (b2) which implies that $t_1 < \textsc{ndl}(d_1)$ and $t_2 = \textsc{ndl}(d_2, \mathbf{x}(t_1+s))$. All the Lagrange multipliers A_i can safely be set to zero and a solution sought in which the partial derivatives (14) and (15) are zero together with

$$\begin{aligned}\frac{\partial Y}{\partial B_I} &= 0 \\ B_i &= 0 \quad \text{for } i \neq I\end{aligned}$$

if they exist within the constraint set. Effectively I is the controlling tissue: the final tissue saturation in tissue I is then equal to the surfacing limit $M_{0,I}$.

There are m possible candidates for the identity of the controlling tissue I. Putting $\partial Y/\partial t_1 = 0$ gives

$$\begin{aligned}0 &= d_1 + \sum_i B_i[-k_i Q_1 e^{-k_i(s+t_1+t_2)}] \\ &= d_1 - Q_1 k_I B_I e^{-k_I(s+t_1+t_2)}\end{aligned}$$

implying

$$k_I B_I Q_1 = d_1 e^{k_I(s+t_1+t_2)}. \tag{16}$$

Moreover $\partial Y/\partial t_2 = 0$ forces

$$\begin{aligned}0 &= d_2 + \sum_i B_i[k_i Q_{1,i}(1-e^{-k_i t_1})e^{-k_i t_2} - k_i Q_2 e^{-k_i t_2}] \\ &= d_2 + B_I[k_I Q_{1,I}(1-e^{-k_I t_1})e^{-k_I t_2} - k_I Q_2 e^{-k_I t_2}] \\ &= d_2 + k_I B_I[Q_{1,I}(1-e^{-k_I t_1})e^{-k_I t_2} - Q_2 e^{-k_I t_2}]\end{aligned}$$

and then substituting (16) gives

$$\begin{aligned}0 &= d_2 + d_1 e^{k_I(t_1+t_2)}\left[1 - e^{-k_I t_1} - \frac{Q_2}{Q_{1,I}}\right] e^{-k_I t_2} \\ &= d_2 + d_1 e^{k_I t_1}\left[1 - e^{-k_I t_1} - \frac{Q_2}{Q_{1,I}}\right] \\ &= d_2 - d_1 + d_1 \frac{Q_{1,I} - Q_2}{Q_{1,I}} e^{k_I t_1}\end{aligned}$$

so that

$$e^{k_I t_1} = \frac{Q_{1,I}(d_1 - d_2)}{d_1(Q_{1,I} - Q_2)}. \tag{17}$$

For $\partial Y/\partial B_I = 0$ then

$$M_{0,I} - Q_0 - Q_{1,I}(1 - e^{-k_I t_1})e^{-k_I t_2} - Q_2(1 - e^{-k_I t_2}) = 0$$

yielding

$$e^{k_I t_2} = \frac{Q_{1,I}(1 - e^{-k_I t_1}) - Q_2}{M_{0,I} - Q_0 - Q_2}.$$

The numerator on the right hand side of the last expression is, using (17),

$$\begin{aligned} Q_{1,I} - Q_2 - Q_{1,I}e^{-k_I t_1} &= Q_{1,I} - Q_2 - Q_{1,I}\frac{d_1(Q_{1,I} - Q_2)}{Q_{1,I}(d_1 - d_2)} \\ &= (Q_{1,I} - Q_2)\left[1 - \frac{d_1}{d_1 - d_2}\right] \\ &= -\frac{(Q_{1,I} - Q_2)d_2}{(d_1 - d_2)} \end{aligned}$$

so that

$$e^{k_I t_2} = -\frac{(Q_{1,I} - Q_2)d_2}{(d_1 - d_2)}\,\frac{1}{M_{0,I} - Q_0 - Q_2}. \tag{18}$$

Equations (17) and (18) together determine at most m candidates for optimal solutions (t_1, t_2) at generic stationary points.

Case (c) If two constraint functions are to be simultaneously zero then one of three possibilities occur: either

(c1) $a_i(t_1) = a_j(t_1) = 0$ for some $i \neq j$; or

(c2) $b_i(t_1, t_2) = b_j(t_1, t_2) = 0$ for some $i \neq j$; or

(c3) $a_i(t_1) = 0$ for some i and $b_j(t_1, t_2) = 0$ for some j,

while all other constraint functions are assumed to take nonzero values. The equality $a_i(t_1) = 0$ implies that $t_1 = \text{NDL}(d_1)$; similarly $b_j(t_1, t_2) = 0$ leads to $t_2 = \text{NDL}(d_2, \mathbf{x}(t_1 + s))$. Of the three possibilities the first (c1) can be excluded because it implies $t_2 < \text{NDL}(d_2, \mathbf{x}(t_1 + s))$. Thus we only need to examine the other two; we remark that (c3) requires that $t_1 = \text{NDL}(d_1)$ and $t_2 = \text{NDL}(d_2, \mathbf{x}(t_1 + s))$ which is the double dive conducted to the no-decompression limits on each occasion.

Subcase (c2) is equivalent to identifying those cases when the no-decompression limit for the second dive is controlled by two tissues; that is, when the minimum in (8) is attained by two tissues i and j, for initial state (9). First consider the single tissue i. Observe that $b_i(t_1, t_2) = 0$ if and only if

$$Q_{1,i}(1 - e^{-k_i t_1})\alpha + Q_2(1 - \alpha) = E_i$$

where $\alpha = e^{-k_i t_2}$ and $E_i = M_{0,i} - Q_0$. A solution with $\alpha \in (0, 1)$ occurs whenever either

$$Q_{1,i}(1 - e^{-k_i t_1}) < E_i < Q_2$$

or

$$Q_{1,i}(1 - e^{-k_i t_1}) > E_i > Q_2.$$

If the set

$$D_i = \{t_1 > 0 : \ b_i(t_1, t_2) = 0 \text{ for some } t_2 > 0\}$$

then solutions with $\alpha \in (0, 1)$ are feasible for all $t_1 > 0$ if $Q_{1,i} \leq E_i \leq Q_2$. In contrast, if $Q_{1,i} \leq E_i$ and $E_i > Q_2$ then no solutions occur whatever the value of t_1. Lastly, $D_i = (0, c_i)$ if $Q_{1,i} > E_i$ and $E_i \leq Q_2$; and $D_i = (c_i, \infty)$ if $Q_{1,i} > E_i > Q_2$, where

$$c_i = -\frac{1}{k_i} \log \left[1 - \frac{E_i}{Q_{1i}} \right].$$

We remark that for any $t_1 \in D_i$ the solution of $b_i(t_1, t_2) = 0$ in t_2 exists and equals

$$f_i(t_1) = \frac{1}{k_i} \log \frac{M_{0i} - Q_0 - Q_2}{Q_{1i}(1 - e^{-k_i t_2}) - Q_2}.$$

Thus, for two tissues i and j, a solution of

$$b_i(t_1, t_2) = b_j(t_1, t_2) = 0$$

exists precisely when the function $g_{ij}(t_1) = f_i(t_1) - f_j(t_1)$ has a root t_1^* in the intersection $D_i \cap D_j$. If this happens then putting $t_2^* = f_i(t_1^*) = f_j(t_1^*)$ yields the required solution (t_1^*, t_2^*).

The implication is that to compute subcase (c2) we examine each pair of tissues i and j in turn. It has to be determined whether the intersection $I_{ij} = D_i \cap D_j$ is nonempty; if this is so we have to find whether a root of g_{ij} exists in I_{ij}. In this eventuality the solution (t_1^*, t_2^*) has to be computed and checks made to ensure that the solution satisfies the remaining constraints, $a_k(t_1^*) \geq 0$ for all k and $b_k(t_1^*, t_2^*) \geq 0$ for all $k \neq i, j$.

4.3. A numerical example

We illustrate the calculation for the optimum double dive taking the parameters used in constructing Figure 2, i.e. successive dives to $d_1 = 40$ and $d_2 = 12$ metres separated by a surface interval of $s = 15$ minutes. We again use the DSAT model parameters summarised in Table 1.

Candidates for the optimum double dive were calculated following the procedure described above. The possibilities are listed in Table 3 which reveals that the best plan is achieved by (b2), the stationary point, with $t_1 = 7.33$ minutes and $t_2 = 130.4$ minutes. For comparison if the first dive is conducted up to its no-decompression limit of $t_1 = 8.94$ minutes (case (c3)), the second dive is then restricted to $t_2 = 125$ minutes. We remark that for case (c2) there is actually a crossing between tissues 5 and 8 at $(t_1, t_2) = (3.19, 188.97)$ but this is an infeasible solution since this value of t_2 exceeds the NDL for the second dive.

Figure 3 shows the maximum value of $\Phi = d_1 t_1 + d_2 t_2$ taken over the possible range of t_2 for each fixed t_1. The optimum clearly occurs at the stationary point of this graph,

case	t_1	t_2	$t_1d_1 + t_2d_2$
(a1)	0	152.8	1833.2
(b2)	7.32	130.4	1858.2
(c3)	8.94	125.0	1856.7

TABLE 3: Candidates for the optimum double dive using the parameters of Figure 2.

where $t_1 \approx 7.32$ minutes. We noted earlier that, to a first approximation, the graph of t_2 against t_1 in Figure 2 is virtually linear with a slope of approximately -3.1. Since $d_2/d_1 = 40/1.2 = 3.33$, the graph of $d_1t_1 + d_2t_2$ against t_1 is also essentially linear with an almost negligible slope. Figures that gives rise to the stationary point in Figure 3.

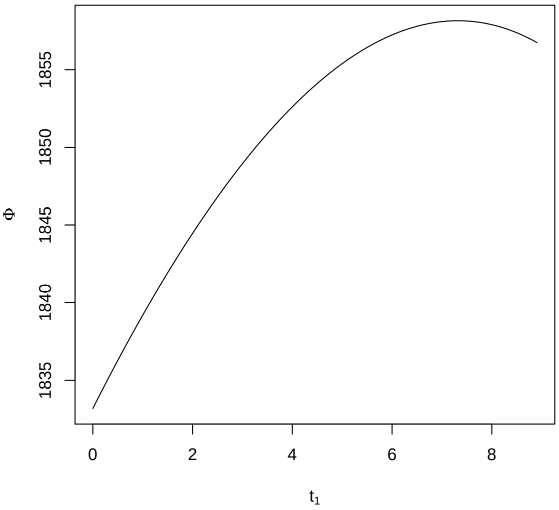

FIGURE 3: Maximised value of $d_1t_1 + d_2t_2$ (maximised over t_2 for fixed t_1) plotted against the duration of first dive t_1. Same parameters as used in Figure 2.

Continuing with the same example, Figure 4 shows the effect of varying the surface interval s. For small values of s (less than about 3 minutes) the optimum is achieved when $t_1 = 0$; informally this arises because the nitrogen build-up is great during the first (deep) dive and the recovery period on the surface very short. Then the reduction in the second dive duration due to the after-effects of the first is so severe that the optimal double dive would forego the first dive altogether. When s is large, that is greater than about 20 minutes, the optimum is achieved when the first dive is extended to its full NDL $t_1 = \textsc{ndl}(d_1)$. For intermediate values of s the optimum typically occurs at a stationary point.

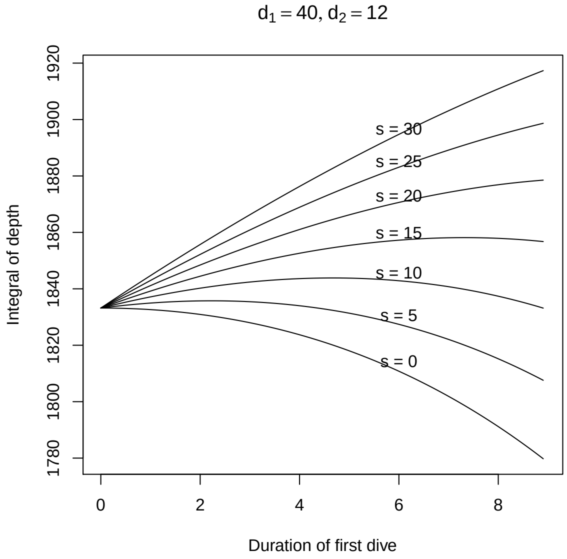

FIGURE 4: Family of curves analogous to previous Figure but allowing for various surface intervals s.

5. Discussion

In this article we have shown how elementary decompression theory can be developed using simple differential equation models. It is seen how optimal dive plans can be deduced, although this issue is not as simple as might have been envisaged at the outset for the reason that the profile of the dive pattern is sensitively dependent on the definition of a good dive. In principle what we have developed is sufficient to formulate a complete set of no-decompression tables for both single and combination dives. Software for performing the calculations in the paper is available [1].

We have deliberately steered clear of discussing decompression dives. There is no technical reason why such dives cannot be handled using exactly the same technology as used here subject to the complication that stops at specified depths would need to be incorporated into the model. Of course there comes a point where the intrinsic attractiveness of analytic formulae is overtaken by the sheer number of free parameters and a numerical solution is then more efficient. Nevertheless, we would argue that the modelling described here is easily extended in many directions and thus ideal for further investigation. Of particular interest might be an examination of the effect on the NDL of imposing specified descent and ascent time-histories and the construction of the best dive plan comprising of more than two individual dives. The modelling of decompression is a topic that contains a richness of possibilities although there is nothing quite as exciting as putting the theory into practice on a sunny day.

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